

# LOCALLY COMPACT HECKE PAIRS

VAHID SHIRBISHEH

**ABSTRACT.** We introduce an extended setting to study Hecke pairs  $(G, H)$  which admit a regular representation on  $L^2(H \backslash G)$ , and consequently a  $C^*$ -algebra. As the result, many pairs of locally compact groups which had been studied in noncommutative harmonic analysis, Lie theory and representation theory are included in the theory of Hecke  $C^*$ -algebras. These Hecke pairs mainly consist of locally compact groups and their compact subgroups, or cocompact subgroups, or open Hecke subgroups. We clarify similarities, differences and relationships of our formulation with the discrete case, and thereby we obtain new results for discrete Hecke pairs too. In the discrete case, using the Schlichting completion and our results, we show that the left regular representations of associated Hecke algebras are bounded homomorphisms. We observe that the relative unimodularity of a discrete Hecke pair amounts to the condition that the left regular representation be a  $*$ -homomorphism. On the other hand, it is shown in [27] that relative unimodularity is a necessary condition for a Hecke pair to possess property (RD). Motivated by these facts, we also give several criteria for relative unimodularity of discrete Hecke pairs.

## 1. INTRODUCTION

Hecke  $C^*$ -algebras were introduced by Jean-Benoît Bost and Alain Connes in [6], in order to study the class field theory of the field  $\mathbb{Q}$  of rational numbers by means of quantum statistical mechanics. Since then, many examples of Hecke pairs  $(G, H)$  have appeared in the related literature. In these Hecke pairs either  $H$  is a subgroup of a discrete group  $G$  or  $H$  is a compact open subgroup of a locally compact group  $G$ . In both cases the necessary and sufficient condition to define a regular representation and subsequently to associate a  $C^*$ -algebra to the pair  $(G, H)$  is that  $H$  should be commensurable with all its conjugates. In this setting the homogeneous space  $H \backslash G$  associated to the pair  $(G, H)$  is always a discrete space, and so we refer to such pairs as discrete Hecke pairs. The discreteness of the homogeneous space  $H \backslash G$  facilitates many constructions and results. However, this setting misses many interesting pairs of locally compact groups and their subgroups which have been studied in other fields, for instance  $(SL_2(\mathbb{R}), SO_2(\mathbb{R}))$  and  $(SL_2(\mathbb{R}), SL_2(\mathbb{Z}))$ . We shall see that these pairs also admit  $C^*$ -algebraic representations similar to the Hecke  $C^*$ -algebras defined by Bost and Connes. On the other hand, even in the discrete case, it was observed in [28, 16] by Kroum Tzanev, S. Kaliszewski, Magnus B. Landstad, and John Quigg that the theory of locally compact groups provides us with a convenient and fruitful setting for studying discrete Hecke pairs and their associated Hecke  $C^*$ -algebras. Another motivation for studying locally compact Hecke pairs comes from our results in the present paper and in our next paper [27], where we apply our generalized setting based on locally compact groups to study length functions on Hecke pairs and property (RD) for locally compact Hecke pairs. Therefore our main aim in this paper is to extend the definition

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of Hecke  $C^*$ -algebras for several classes of pairs  $(G, H)$  of locally compact groups whose homogeneous spaces  $H \backslash G$  are not necessarily discrete.

In this paper, a pair  $(G, H)$  consisting of a locally compact group  $G$  and a closed subgroup  $H$  of  $G$  is simply called a *pair*. Regarding the Bost-Connes construction of Hecke  $C^*$ -algebras, we carefully examine various steps of associating a  $C^*$ -algebra to a “suitable” pair. Our primary concern is to determine classes of pairs such that for every pair  $(G, H)$  in these classes, there exists a left regular representation from the Hecke algebra  $\mathcal{H}(G, H)$  associated to  $(G, H)$  into the  $C^*$ -algebra of bounded operators on the Hilbert space  $L^2(H \backslash G)$ . Besides discrete Hecke pairs, we show that there are at least two other important classes of pairs which admit such a construction, and therefore their Hecke algebras are suitable for  $C^*$ -algebraic formulation. The first class consists of pairs  $(G, H)$  in which  $H$  is a compact subgroup of  $G$ . In this case, not only we prove the existence of a well defined left regular representation, but we also show that it is a bounded homomorphism, see Theorem 4.1. Although this latter statement is not a necessary step to define Hecke  $C^*$ -algebra, it is a useful feature in application, due to the parallelism between reduced Hecke  $C^*$ -algebras and reduced group  $C^*$ -algebras. Furthermore, by applying a certain topologization process of discrete Hecke pairs named the Schlichting completion, this statement is applied to prove the same boundedness statement for the left regular representations associated with discrete Hecke pairs. This is particularly important in the study of property (RD) for Hecke pairs, for instance it is applied to show every locally compact Hecke pair with polynomial growth has property (RD), see Section 4 of [27] for details. The second class of locally compact Hecke pairs capable of a  $C^*$ -algebraic representation consists of pairs  $(G, H)$  in which  $H$  is a cocompact (or at least cofinite volume) subgroup of  $G$ , see Theorem 6.1.

Given a discrete Hecke pair  $(G, H)$ , it is known that a group homomorphism  $\Delta_{(G, H)} : G \rightarrow \mathbb{Q}^+$  is associated to the Hecke pair, which we name it “the relative modular function”. In a sense, it is a generalization of the modular function of a locally compact group. Therefore we define the involution on the Hecke algebra associated with  $(G, H)$  using its relative modular function. Then it follows that the associated left regular representation is an involutive homomorphism if and only if the relative modular function is the constant function 1. Moreover, in Section 3 of [27], we have proved that a necessary condition for a discrete Hecke pair  $(G, H)$  to possess property (RD) is that the relative modular function must be the trivial homomorphism, i.e.  $\Delta_{(G, H)} = 1$ . It is in line with a theorem of R. Ji and L.B. Schweitzer in [15] which asserts that if a locally compact group  $G$  has property (RD), then it must be unimodular. This is one of the evidences that even discrete Hecke pairs have some common features with locally compact groups. Motivated by these observations, we discuss various algebraic and analytic criteria to determine when the relative modular function of a discrete Hecke pair  $(G, H)$  is trivial. For instance, it is shown that if the Hecke algebra associated to  $(G, H)$  satisfies a certain condition which is weaker than commutativity, then the relative modular function is trivial, see Proposition 3.4.

Following the works of G. Schlichting in [21, 22], K. Tzanev, S. Kaliszewski, Magnus B. Landstad, and John Quigg in [28, 16], constructed a certain procedure for densely embedding a discrete (reduced) Hecke pair  $(G, H)$  into a Hecke pair  $(\overline{G}, \overline{H})$ , where  $\overline{G}$  is a totally disconnected locally compact group and  $\overline{H}$  is a compact open subgroup of  $\overline{G}$ . This technique is called the Schlichting completion of Hecke pairs and it is our main tool

to transfer several results from locally compact groups and Hecke pairs into the setting of discrete Hecke pairs. Besides applying the Schlichting completion in the proof of our results, we also discuss several of its examples in Example 5.2.

The works presented in this paper and its companion [27] not only contribute to the theory of Hecke  $C^*$ -algebras and property (RD) for Hecke pairs, but they also serve as the prototypes of ideas and methods which relate the study of Hecke pairs with the area of locally compact (and discrete) groups. Therefore it is expected that these ideas will be expanded to study other aspects related to these two areas. There are many other interesting subjects to be studied yet, such as the geometric theory of Hecke pairs, the role of Hecke pairs in induced representations, constructing a  $C^*$ -algebraic framework to study Gelfand pairs, etc. We hope that this paper provides sufficient preparations and clues for further developments in the study of these fascinating subjects.

## 2. LOCALLY COMPACT HECKE PAIRS $(G, H)$

Our main aim in this section is to define an extended setting to study Hecke pairs  $(G, H)$  which admit left regular representations on  $L^2(H \backslash G)$ . Here, our guiding principle in developing the theory of locally compact Hecke pairs is that it should coincide with the discrete case when the subgroup in the Hecke pair is an open Hecke subgroup. In this way, we will be able to transfer useful results and ideas from locally compact case into discrete case and vice versa. In order to define a general and consistent  $C^*$ -algebraic framework to study Hecke algebras, we need to define both an algebra and an involution on this algebra. These two steps are handled slightly differently in discrete and non-discrete cases. So, we should differ between these cases.

**Definition 2.1.** (i) Let  $(G, H)$  be a pair of locally compact groups. It is called a *discrete pair*, if  $H$  is open in  $G$ , otherwise it is called *non-discrete*.  
 (ii) A discrete pair  $(G, H)$  is called a *discrete Hecke pair* if every double coset of  $H$  in  $G$  is a finite union of finitely many left cosets of  $H$  in  $G$ . In this case, we also say that  $H$  is a *Hecke subgroup of  $G$* <sup>1</sup>.  
 (iii) Given a discrete Hecke pair  $(G, H)$ , the vector space of all finite support complex functions on the set  $G // H$  of double cosets of  $H$  in  $G$  is denoted by  $\mathcal{H}(G, H)$  and the above condition allows us to define a *convolution product* on  $\mathcal{H}(G, H)$  by

$$(2.1) \quad f_1 * f_2(HxH) := \sum_{Hy \in H \backslash G} f_1(Hxy^{-1}H) f_2(HyH),$$

for all  $f_1, f_2 \in \mathcal{H}(G, H)$  and  $x \in G$ .

Given a discrete Hecke pair, the vector space  $\mathcal{H}(G, H)$  with the above product is a complex algebra, which is usually called the Hecke algebra associated with the pair  $(G, H)$ , see [13, 17] for detailed introductions to this type of Hecke algebras. The algebra  $\mathcal{H}(G, H)$  also admits an involution defined by

$$(2.2) \quad f^\circ(HxH) := \overline{f(Hx^{-1}H)}, \quad \forall f \in \mathcal{H}(G, H), x \in G.$$

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<sup>1</sup>Alternative names for Hecke subgroups appearing in the literature are “almost normal subgroups” and “commensurated subgroups”, see [6, 16, 24]. However, we follow [16]. See also Example 3.5 below for the origin of the name “Hecke subgroup”. Besides, we use the phrase “almost normal” for another notion, see Definition 3.9(iii) below.

However, due to some technical reasons explained below, we have to consider a slightly different involution on  $\mathcal{H}(G, H)$ , see Definition 2.5.

Using a convolution product similar to (2.1),  $\mathcal{H}(G, H)$  is endowed with a *left regular representation* as follows:

$$(2.3) \quad \begin{aligned} \lambda : \mathcal{H}(G, H) &\rightarrow B(\ell^2(H \backslash G)), \\ \lambda(f)(\xi)(Hx) &:= (f * \xi)(Hx), \end{aligned}$$

for all  $f \in \mathcal{H}(G, H)$ ,  $\xi \in \ell^2(H \backslash G)$  and  $Hx \in H \backslash G$ . By definition, the *reduced Hecke  $C^*$ -algebra of the Hecke pair  $(G, H)$*  is the norm completion of the image of this representation. The class of these Hecke pairs also includes pairs  $(G, H)$ , where  $G$  is a locally compact group and  $H$  is a compact open subgroup of  $G$ . These Hecke pairs and their associated Hecke  $C^*$ -algebras have been studied extensively in the literature, see for instance [8, 13, 16, 28].

Now we turn our attention to non-discrete pairs. As it is clear from the discrete case, we are not only interested in the Hecke algebra of a Hecke pair  $(G, H)$ , but also we need an explicit regular representation to embed this algebra inside the  $C^*$ -algebra of bounded operators on the Hilbert space  $L^2(H \backslash G)$ , where the homogeneous space  $H \backslash G$  is equipped with an appropriate measure. This requirement makes us to define the convolution product,  $L^1$ -norm and  $L^2$ -norm using integrations over the locally compact space of right cosets of  $H$  in  $G$ , and therefore, we need to set up an appropriate measure theoretic framework. In what follows, we often follow the notations of Folland's book [11] for measure theoretic considerations.

In this paper,  $\mu$  always denotes a right Haar measure on  $G$ ,  $\eta$  denotes a right Haar measure on  $H$ ,  $\Delta_G$  and  $\Delta_H$  denote the modular functions of  $G$  and  $H$ , respectively. The vector space of all complex valued compact support continuous functions on a (locally compact) topological space  $X$  is denoted by  $C_c(X)$ . There is an onto map  $P : C_c(G) \rightarrow C_c(H \backslash G)$  defined by

$$Pf(Hx) := \int_H f(hx) d\eta(h), \quad \forall f \in C_c(G), x \in G.$$

It is well known that we have

$$(2.4) \quad \Delta_G|_H = \Delta_H$$

if and only if there exists a right  $G$ -invariant Radon measure  $\nu$  on  $H \backslash G$ . In this case,  $\nu$  is unique up to a positive multiple. By choosing the multiple suitably, we get *Weil's formula* for the decomposition of an integral on  $G$  into a double integral on  $H$  and  $H \backslash G$  as follows:

$$(2.5) \quad \int_G f(x) d\mu(x) = \int_{H \backslash G} Pf(y) d\nu(y) = \int_{H \backslash G} \int_H f(hy) d\eta(h) d\nu(y),$$

for all  $f \in C_c(G)$ . Then we briefly say that *the triple  $(\eta, \mu, \nu)$  satisfies Weil's formula*. One notes that  $y$  in the above formula varies over a complete set of representatives of right cosets. In fact, we could use  $Hy$  instead of  $y$ , and we sometimes use this latter notation in order to clarify our computations.

**Assumption 2.2.** We restrict our study to those pairs  $(G, H)$  whose homogeneous spaces  $H \backslash G$  possess right  $G$ -invariant Radon measures, denoted by  $\nu$ , satisfying Equality (2.5).

Inspired by the discrete case, for a given non-discrete pair  $(G, H)$ , we define

$$(2.6) \quad \mathcal{H}(G, H) := \{f \in C_c(H \backslash G); f(Hxh) = f(Hx), \forall x \in G, h \in H\}.$$

Every element of  $\mathcal{H}(G, H)$  can be thought of as a function on the set of double cosets of  $H$  in  $G$ . Therefore, for every  $f \in \mathcal{H}(G, H)$  and  $x \in G$ , each of the expressions  $f(HxH)$ ,  $f(xH)$  and  $f(x)$  has the same meaning as  $f(Hx)$ . The complex vector space  $\mathcal{H}(G, H)$  is equipped with the following convolution product:

$$(2.7) \quad f * g(Hx) := \int_{H \backslash G} f(Hxy^{-1}H)g(Hy)d\nu(Hy), \quad \forall f, g \in \mathcal{H}(G, H).$$

For  $f$  and  $g$  as above, let  $S_f$  and  $S_g$  be the supports of  $f$  and  $g$ , respectively. Let  $\pi : G \rightarrow H \backslash G$  be the natural quotient map. By Lemma 2.47 of [11], there are compact subsets  $A_f, A_g \subseteq G$  such that  $\pi(A_f) = S_f$  and  $\pi(A_g) = S_g$ . Since  $A_f A_g$  is a compact subset of  $G$ ,  $\pi(A_f A_g)$  is compact, and one easily observes that  $\text{supp}(f * g) \subseteq \pi(A_f A_g)$ . It is straightforward to check that  $f * g$  is continuous. Finally, it follows from the right  $G$ -invariance of  $\nu$  that  $f * g \in \mathcal{H}(G, H)$ . Thus the above product is well defined.

In order to define appropriate involutions on the Hecke algebra  $\mathcal{H}(G, H)$ , again we have to treat discrete and non-discrete cases, separately. We proceed with some notations for the discrete case. For every discrete Hecke pair  $(G, H)$ , we define two functions  $L, R : G \rightarrow \mathbb{N}$  by

$$L(x) := [H : H_x] = |HxH/H|, \quad R(x) := [H : H_{x^{-1}}] = |H \backslash HxH|,$$

for all  $x \in G$ , where  $H_x := H \cap xHx^{-1}$ . The integer  $L(x)$  (resp.  $R(x)$ ) is the number of distinct left (resp. right) cosets appearing in the double coset  $HxH$ , and so we have  $R(x) = L(x^{-1})$ .

**Definition 2.3.** By definition, the *relative modular function of a discrete Hecke pair*  $(G, H)$  is the function  $\Delta_{(G,H)} : G \rightarrow \mathbb{Q}^+$  defined by

$$\Delta_{(G,H)}(x) := \frac{L(x)}{R(x)}, \quad \forall x \in G.$$

The discrete Hecke pair  $(G, H)$  is called *relatively unimodular* if  $\Delta_{(G,H)}(x) = 1$  for all  $x \in G$ .

In fact, the function  $\Delta_{(G,H)}$  is a group homomorphism whose kernel contains  $H$ , see Theorem 2.2 of [28]. It follows that  $\Delta_{(G,H)}$  is always a continuous bi- $H$ -invariant function on  $G$ . Moreover, when  $H$  is a compact open subgroup of  $G$ , it was noted in [28, 16] that

$$(2.8) \quad \Delta_{(G,H)}(x) = \Delta_G(x), \quad \forall x \in G,$$

see for instance Page 669 of [16] for a proof. We should also mention that the above equality also appeared in Lemma 1 of G. Schlichting's paper [21].

It is important to note that the left regular representation  $\lambda$ , defined in (2.3), is an involutive homomorphism with respect to the involution defined in (2.2). However, this involution does not necessarily preserve the  $\ell^1$ -norm of elements of  $\mathcal{H}(G, H)$ . Even worse, one can see that this involution is not always continuous with respect to the  $\ell^1$ -norm. Therefore we are not able to show directly that the left regular representation  $\lambda$  is continuous. The following example illustrates these points:

**Example 2.4.** Let  $(G, H)$  be the Bost-Connes Hecke pair, that is

$$G = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}; a \in \mathbb{Q}^+, b \in \mathbb{Q} \right\}, \quad \text{and} \quad H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}; n \in \mathbb{Z} \right\}.$$

For given  $g = \begin{pmatrix} 1 & b \\ 0 & \frac{m}{n} \end{pmatrix} \in G$ , where  $m$  and  $n$  have no common prime factors, one easily computes  $L(g) = n$  and  $R(g) = m$ , see 2.1.1.3 of [13]. Let  $\chi_g$  denote the characteristic function of the double coset  $HgH$  considered as an element of  $\mathcal{H}(G, H)$ . Then  $\|\chi_g\|_1 = R(g) = m$  and  $\|\chi_g^{\otimes}\|_1 = L(g) = n$ . Therefore by replacing  $g$  with the elements of the sequence  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \right\}_{n \in \mathbb{N}}$ , we observe that the involution  $^{\otimes}$  on  $\mathcal{H}(G, H)$  is not continuous in  $\ell^1$ -norm.

The problem is that the mapping  $Hx \mapsto Hx^{-1}$  is not even a well defined change of variable in  $H \backslash G$ , of course, unless  $H$  is a normal subgroup of  $G$ . However, the map  $HxH \mapsto Hx^{-1}H$  is a well defined bijection over the set of double cosets of  $H$  in  $G$  and so the involution  $^{\otimes}$  is well defined. To obtain a norm preserving involution which can be generalized to non-discrete case, we use the definition given in [16]:

**Definition 2.5.** For a discrete Hecke pair  $(G, H)$ , the following involution is defined on the algebra  $\mathcal{H}(G, H)$ :

$$(2.9) \quad f^*(HxH) := \Delta_{(G,H)}(x^{-1}) \overline{f(Hx^{-1}H)}, \quad \forall f \in \mathcal{H}(G, H), x \in G.$$

In the following example, we observe that the same definition for relative modular functions on non-discrete pairs is not possible:

**Example 2.6.** Let  $G$  be the special linear group of degree 2,  $SL_2(\mathbb{R})$ , and let  $H$  be its compact subgroup  $SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}; -\pi < \theta \leq \pi \right\}$ . For a non-zero real number  $t$ , set  $x := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ . Then one computes that  $H_x = \{I, -I\}$ , where  $I$  is the  $2 \times 2$  identity matrix. Therefore  $[H : H_x] = \infty$ . This shows that functions  $R$ ,  $L$  and  $\Delta_{(G,H)}$  are not well defined for the pair  $(G, H)$ , but we shall see that the pair  $(G, H)$  admits a left regular representation, and so it will be considered as a (non-discrete) Hecke pair in our generalized setting.

Due to the lack of relative modular functions on non-discrete Hecke pairs, Formula (2.9) cannot be adopted for the non-discrete case. However, Equality (2.8) suggests that we use the modular function of  $G$  in the non-discrete case in lieu of the relative modular function in the discrete case. In order to this work, we have to impose another restriction:

**Assumption 2.7.** When  $(G, H)$  is a non-discrete pair, we assume  $H$  is unimodular.

Thus, regarding the above assumption and Assumption 2.2, for a non-discrete Hecke pair  $(G, H)$ , we always assume  $\Delta_G|_H = \Delta_H = 1$ . One notes that when  $H$  is compact, this condition is automatically satisfied. Moreover, when there is no risk of confusion, we can ignore  $\Delta_H$  and denote the modular function of  $G$  simply by  $\Delta$ . It follows from the above assumption that  $\Delta$  is a bi- $H$ -invariant function on  $G$ , so the following definition is allowed:

**Definition 2.8.** Assume  $(G, H)$  is a non-discrete pair. The involution on the algebra  $\mathcal{H}(G, H)$  is defined by

$$(2.10) \quad f^*(Hx) := \Delta(x^{-1}) \overline{f(Hx^{-1})}, \quad \forall f \in \mathcal{H}(G, H), x \in G.$$

To see that how the right  $G$ -invariance of  $\nu$  is used to prove that this map is an involution on  $\mathcal{H}(G, H)$ , we show that it is an anti-homomorphism. The rest of axioms of involution are straightforward, and so are left to the reader. For every  $f, g \in \mathcal{H}(G, H)$  and  $x \in G$ , we compute

$$(f * g)^*(Hx) = \Delta(x^{-1}) \overline{f * g(Hx^{-1})} = \Delta(x^{-1}) \overline{\int_{H \setminus G} f(x^{-1}y^{-1})g(y)d\nu(y)}$$

Set  $Hx := Hyx$ . Due to the right  $G$ -invariance of  $\nu$ , this is a measure preserving change of variable over  $H \setminus G$ . Thus we have

$$\begin{aligned} (f * g)^*(Hx) &= \int_{H \setminus G} \Delta(x^{-1}) \overline{f(z^{-1})} \overline{g((xz^{-1})^{-1})} d\nu(z) \\ &= \int_{H \setminus G} \Delta(x^{-1}) \Delta(z) f^*(z) \Delta(xz^{-1}) g^*(xz^{-1}) d\nu(z) = g^* * f^*(Hx). \end{aligned}$$

Now that we have defined appropriate involutions for both discrete and non-discrete cases, we are ready to define Hecke algebras:

**Definition 2.9.** (i) When  $(G, H)$  is a discrete Hecke pair, the algebra  $\mathcal{H}(G, H)$  of finite support complex function on the set of double cosets of  $H$  in  $G$  with the involution defined in (2.9) is called the *Hecke algebra of the discrete Hecke pair*  $(G, H)$ .  
 (ii) When  $(G, H)$  is a non-discrete pair satisfying Assumptions 2.2 and 2.7, the involutive algebra  $\mathcal{H}(G, H)$  defined by (2.6), (2.7) and (2.10) is called the *Hecke algebra of the pair*  $(G, H)$ .

We have not defined a Hecke pair in the non-discrete case yet. The reason is that unlike discrete cases, a non-discrete pair  $(G, H)$  satisfying Assumptions 2.2 and 2.7, and without any extra conditions, gives rise to a Hecke algebra. In fact, part of the condition of  $H$  being a Hecke subgroup of  $G$  is encoded in the definition of  $\mathcal{H}(G, H)$  in (2.6), see also Remark 2.11(i). Since our intention is to study Hecke algebras in the realm of  $C^*$ -algebras, we seek those conditions which facilitate the embedding of Hecke algebras inside a  $C^*$ -algebra. Unfortunately, we are not always able to define a left regular representation on the Hilbert space  $L^2(H \setminus G)$ , as it is clear in the discrete case, see Remark 2.12. On the other hand, the requirement that every double coset has only finitely many left cosets is not necessary for a non-discrete pair  $(G, H)$  to admit a left regular representation, see Example 2.6 and Theorem 4.1. Therefore we propose the following definition as the generalization of Definition 2.1(ii) for non-discrete pairs  $(G, H)$ :

**Definition 2.10.** (i) Let  $(G, H)$  be a non-discrete pair satisfying Assumptions 2.2 and 2.7. It is called a *non-discrete Hecke pair* if the followings hold:  
 (a) The mapping  $\lambda : \mathcal{H}(G, H) \rightarrow B(L^2(H \setminus G, \nu))$  defined by  $\lambda(f)(\xi) := f * \xi$  for all  $f \in \mathcal{H}(G, H)$  and  $\xi \in L^2(H \setminus G)$  is a well defined homomorphism.  
 (b) For every  $x \in G$ , there exists some  $f \in \mathcal{H}(G, H)$  such that  $f(Hx) \neq 0$ .  
 (ii) When  $(G, H)$  is a non-discrete Hecke pair, the homomorphism  $\lambda$  is called the *left regular representation of the Hecke pair*  $(G, H)$  and the norm completion of the image of  $\mathcal{H}(G, H)$  under  $\lambda$  in the  $C^*$ -algebra  $B(L^2(H \setminus G))$  is called the *reduced Hecke  $C^*$ -algebra of the Hecke pair*  $(G, H)$  and is denoted by  $C_r^*(G, H)$ .

**Remark 2.11.** (i) To see the relevance of Condition (b) of Definition 2.10(i), let  $H$  be an arbitrary subgroup of a discrete group  $G$  and define the *commensurator* of

$H$  in  $G$  by  $Comm_G(H) := \{g \in G; R(g), L(g) < \infty\}$ . It is known that  $Comm_G(H)$  is a subgroup of  $G$  and  $Comm_G(H) = G$  if and only if  $H$  is a Hecke subgroup of  $G$ . The vector space  $\mathcal{H}(G, H)$  as defined in (2.6) coincides with the vector space of all finite support complex functions on the set  $G//H$  of double cosets of  $H$  in  $G$  if and only if Condition (b) holds. Otherwise, the supports of functions in  $\mathcal{H}(G, H)$  would be contained in  $Comm_G(H)//H$ . Condition (b) is especially important in the non-discrete case that we do not have any algebraic condition to impose the same restriction.

- (ii) One notes that in the case that  $G$  is a discrete group or the case that  $H$  is a compact open subgroup of a locally compact group  $G$ , Definition 2.10(i) is equivalent to Definition 2.1(ii). Therefore in these cases, the pair  $(G, H)$  can be considered both as discrete and non-discrete. Ambiguity arise only when  $H$  is an open Hecke subgroup of  $G$ . Because, in this case the topology and measure structure of  $G$  and  $H$  are ignored (both are considered discrete), and so we do not have to worry about Assumptions 2.2 and 2.7. That is why we have separated the definition of discrete and non-discrete Hecke pairs.
- (iii) One observes that when  $H$  is a compact or cocompact subgroup of a locally compact group  $G$ , Condition (b) readily holds for the pair  $(G, H)$ .

Assume  $(G, H)$  is a non-discrete Hecke pair satisfying Condition (b) of Definition 2.10(i). To check Condition (a), we must first prove that  $\lambda(f)$  is a bounded operator on the Hilbert space  $L^2(H \backslash G, \nu)$  for all  $f \in \mathcal{H}(G, H)$ . Next, the usual argument, based on the Fubini theorem, shows that  $\lambda$  is a homomorphism, and finally it is clear that  $\lambda$  is always injective.

Regarding the similarity between Hecke pairs and groups, one wishes to prove that  $\lambda$  is a bounded operator, that is to find a constant  $M > 0$  such that  $\lambda(f) \leq M\|f\|_1$ . Although we are not able to prove this last statement for general Hecke pairs, we shall prove it in some important cases, see Theorem 4.1, Theorem 5.4 and Remark 6.3. One also notes that Assumption 2.7 is needed only if we are interested to have an involution on the Hecke algebra  $\mathcal{H}(G, H)$ .

**Remark 2.12.** Assume  $G$  is a discrete group, as it has already been studied in the literature, the pair  $(G, H)$  is a Hecke pair in the sense of Definition 2.10(i) if and only if it is a Hecke pair in the sense of Definition 2.1(ii). The “if” part was proved in Proposition 1.3.3 of [8]. To see the “only if” part, assume there is a double coset  $HxH$  which contains infinitely many distinct right cosets, say  $\{Hx_k\}_{k=1}^\infty$ . For every  $k \in \mathbb{N}$ , let  $\chi_k \in \mathcal{H}(G, H)$  be the characteristic function of the double coset  $Hx_kH$  and let  $\delta_e \in \ell^2(H \backslash G)$  be the characteristic function of the right coset  $H$ . Then one easily computes  $\chi_k * \delta_e(Hx_k) = 1$  for all  $k \in \mathbb{N}$ , and so  $\|\chi_k * \delta_e\|_2 = \infty$ .

The following proposition is a useful tool to realize more pairs  $(G, H)$  of locally compact groups as Hecke pairs. It is also applied in the reduction of Hecke pairs.

**Proposition 2.13.** *Let  $(G, H)$  be a pair and let  $N$  be a normal closed subgroup of  $G$  contained in  $H$ . Set  $G' := \frac{G}{N}$  and  $H' := \frac{H}{N}$ .*

- (i) *If the pair  $(G, H)$  satisfies Assumptions 2.2 and 2.7, then the pair  $(G', H')$  satisfies them too.*
- (ii) *If  $N$  is unimodular and  $H'$  is compact, then both pairs  $(G', H')$  and  $(G, H)$  satisfy Assumptions 2.2 and 2.7.*



- (iii) Assume that the pair  $(G, H)$  satisfies Assumptions 2.2 and 2.7, then the pair  $(G, H)$  is a Hecke pair if and only if the pair  $(G', H')$  is a Hecke pair. In this case, the Hecke algebras  $\mathcal{H}(G, H)$  and  $\mathcal{H}(G', H')$  are isomorphic.
- (iv) With the assumptions of item (iii), the left regular representation of  $\mathcal{H}(G, H)$  is bounded if and only if the left regular representation of  $\mathcal{H}(G', H')$  is bounded. Moreover, the  $C^*$ -algebras  $C_r^*(G, H)$  and  $C_r^*(G', H')$  are isomorphic.

*Proof.* We only deal with the case that  $(G, H)$  is a non-discrete pair. The proof for discrete pairs is way easier, because it does not involve measure theory.

- (i) Let  $(G, H)$  satisfies Assumptions 2.2 and 2.7. Since  $H$  is unimodular and  $N$  is normal in  $H$ , both  $N$  and  $H'$  are unimodular too, and so the assumption 2.7 holds for  $(G', H')$ .

To check Assumption 2.2, it is enough to show that  $H' \backslash G'$  possesses a right  $G'$ -invariant measure. In the following argument  $x$  is always an arbitrary element of  $G$ . Let  $\pi : G \rightarrow G', x \mapsto \bar{x}$  be the quotient map. Define  $\pi' : H \backslash G \rightarrow H' \backslash G'$  by  $Hx \mapsto H'\bar{x}$ . It is easy to see that  $\pi'$  is a well defined bijection. Consider the diagram

$$(2.11) \quad \begin{array}{ccc} G & \xrightarrow{\pi} & G' \\ \downarrow & & \downarrow \\ H \backslash G & \xrightarrow{\pi'} & H' \backslash G' \end{array}$$

Since  $\pi$  and the vertical arrows are continuous open mappings,  $\pi'$  is a homeomorphism. Therefore it gives rise to a linear isomorphism

$$\varphi : C_c(H \backslash G) \rightarrow C_c(H' \backslash G'), \quad \varphi(f)(H'\bar{x}) := f(Hx),$$

for all  $f \in C_c(H \backslash G)$  and  $H'\bar{x} \in H' \backslash G'$ . In the following  $g$  represents an arbitrary element of  $C_c(H' \backslash G')$ . First we note that the inverse of  $\varphi$  is given by  $\varphi^{-1}(g) = g \circ \pi'$ . Next, we define a functional  $I : C_c(H' \backslash G') \rightarrow \mathbb{C}$  by

$$I(g) := \int_{H \backslash G} g \circ \pi'(Hx) d\nu(x).$$

One checks that  $I$  is a positive linear functional on  $C_c(H' \backslash G')$ . Furthermore,  $I(g) = 0$  if and only if  $g = 0$  almost everywhere. It is straightforward to check that  $I$  is invariant by the action of  $G'$  on  $C_c(H' \backslash G')$  induced by the right action of  $G'$  on  $H' \backslash G'$ , that is  $I(R_{\bar{y}}(g)) = I(g)$  for all  $\bar{y} \in G'$ , where as usual,  $R_{\bar{y}}(g)(H'\bar{x}) = g(H'\bar{x}\bar{y})$  for all  $H'\bar{x} \in H' \backslash G'$ . Therefore  $I$  defines a right  $G'$ -invariant Radon measure  $\theta$  on  $H' \backslash G'$ , which amounts to saying that  $\Delta_{G'}|_{H'} = \Delta_{H'}$ .

- (ii) Assumptions 2.2 and 2.7 for  $(G', H')$  easily follow from Proposition 2.27 of [9]. Thus we only need to prove these assumptions for the pair  $(G, H)$ . Since  $N$  is unimodular,  $N \subseteq \text{Ker} \Delta_G$ . Therefore there is a continuous group homomorphism  $\rho : G' \rightarrow \mathbb{R}^+$  such that  $\rho \circ \pi = \Delta_G$ . Since  $H'$  is a compact subgroup of  $G'$ ,  $\rho(H')$  is a compact subgroup of  $\mathbb{R}^+$ . But the only compact subgroup of  $\mathbb{R}^+$  is the trivial subgroup  $\{1\}$ . Hence  $\rho|_{H'} = 1$  and this implies that  $\Delta_G|_H = 1$ . Similarly one shows that  $\Delta_H = 1$ .
- (iii) Regarding the way we defined the measure  $\theta$  on  $H' \backslash G'$  in the above, we have

$$\int_{H' \backslash G'} \varphi(f)(H'\bar{x}) d\theta(H'\bar{x}) = \int_{H \backslash G} f(Hx) d\nu(Hx),$$

for all  $f \in C_c(H \backslash G)$ . Hence the isomorphism  $\varphi$  extends to an isometric isomorphism  $\varphi_2$  between  $L^2(H \backslash G)$  and  $L^2(H' \backslash G')$ . On the other hand, one easily checks that  $\varphi$  maps  $\mathcal{H}(G, H)$  onto  $\mathcal{H}(G', H')$ . Let  $\varphi_{\mathcal{H}} : \mathcal{H}(G, H) \rightarrow \mathcal{H}(G', H')$  denote the linear isomorphism obtained by restricting  $\varphi$  to  $\mathcal{H}(G, H)$ . Then a straightforward computation shows that  $\varphi_{\mathcal{H}}$  preserves the convolution, so it is an algebra isomorphism, and we have

$$\varphi_{\mathcal{H}}(f) * \varphi_2(\xi) = \varphi_2(f * \xi), \quad \forall f \in \mathcal{H}(G, H), \xi \in L^2(H \backslash G).$$

This proves Condition (a) of Definition 2.10(i). Condition (b) follows from the fact that the mapping  $\varphi$  is a homeomorphism.

- (iv) Similar to the above item,  $\varphi$  extends to an isometric isomorphism between  $L^1(H \backslash G)$  and  $L^1(H' \backslash G')$ . Then the desired statements follow immediately from (iii).  $\square$

**Remark 2.14.** With the notations of the above proposition, we note the followings:

- (i) To see that the compactness of  $H'$  is necessary in Proposition 2.13(ii), consider the group  $H = \mathbb{R} \rtimes \mathbb{R}^\times$  and its normal subgroup  $N = \mathbb{R}$ . The locally compact group  $H$  is not unimodular. However, both  $N$  and  $\frac{H}{N} = \mathbb{R}^\times$  are unimodular.
- (ii) It is straightforward to see that the map  $\varphi_{\mathcal{H}}$  preserves involution if and only if  $\Delta_G(x) = \Delta_{G'}(\bar{x})$  for all  $x \in G$ . However, the above example also shows that this formula is not generally true.
- (iii) Fix a Haar measure  $\sigma$  on  $N$ . Let  $\alpha$  and  $\beta$  be the right invariant Haar measures on  $G'$  and  $H'$ , respectively, such that the triples  $(\sigma, \mu, \alpha)$  and  $(\sigma, \eta, \beta)$  satisfy Weil's formula. Then one checks that the triple  $(\beta, \alpha, \theta)$  satisfies Weil's formula.

There are also other ways of constructing new Hecke pairs from given ones. For example, direct product of finitely many Hecke pairs or considering extensions of groups by Hecke pairs, as it was done for discrete Hecke pairs in Section 3 of [25].

### 3. UNIMODULARITY AND RELATIVE UNIMODULARITY

In this section,  $(G, H)$  is always a discrete Hecke pair. Two involutions  $^*$  and  $^\circ$  defined on  $\mathcal{H}(G, H)$  in (2.2) and (2.9), respectively, agree if and only if the Hecke pair  $(G, H)$  is relatively unimodular. On the other hand, it is easy to check that this latter condition holds if and only if the involution  $^\circ$  preserves the  $\ell^1$ -norm of  $H \backslash G$ , see also Example 2.4. Moreover, when  $H$  is a compact open subgroup of a locally compact group  $G$ , the equality  $\Delta_{(G, H)} = 1$  is equivalent to the unimodularity of  $G$ . Also, we will show that the relative unimodularity is a necessary condition for property (RD) of discrete Hecke pairs, see Corollary 3.13 of [27]. Thus it is important to find various criteria implying this condition, which is the subject of the present section. We shall also briefly address the non-discrete pairs  $(G, H)$  for which  $G$  is unimodular. We begin our investigation with some easy observations.

Given  $f \in \mathcal{H}(G, H)$ , to have  $\|f^\circ\|_1 = \|f\|_1$ , an obviously sufficient condition is that  $|f(x)| = |f(x^{-1})|$  for all  $x \in G$ . This condition holds for all  $f \in \mathcal{H}(G, H)$  whenever

$$(3.1) \quad HxH = Hx^{-1}H, \quad \forall x \in G.$$

**Example 3.1.** (i) Consider the Hecke pair  $(SL_2(\mathbb{Q}_p), SL_2(\mathbb{Z}_p))$ , where  $p$  is a prime number. The group  $SL_2(\mathbb{Q}_p)$  is a totally disconnected locally compact group and

$SL_2(\mathbb{Z}_p)$  is a compact open subgroup of this group. Condition (3.1) was verified for this Hecke pair in Section 2.2.3.2 of [13].

- (ii) Let  $G$  be an abelian group. Consider the action  $\theta$  of  $\mathbb{Z}/2$  on  $G$  by inversion, that is  $\theta_{-1}(g) := -g$  for all  $g \in G$ , where  $\mathbb{Z}/2$  has been considered as the multiplicative group  $\{1, -1\}$ . One easily checks that Condition (3.1) holds for every Hecke pair of the form  $(G \rtimes \mathbb{Z}/2, \{0\} \rtimes \mathbb{Z}/2)$ .

Another important consequence of Condition (3.1) is that  $\mathcal{H}(G, H)$  is a commutative algebra, see Section 2.2.3.2 of [13] for a proof. The following example shows that the converse is not true in general. However, we can prove Proposition 3.4 below, which gives a more general condition implying the relative unimodularity.

**Example 3.2.** Consider the semi-crossed product group  $G = (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$  defined by the action of  $\mathbb{Z}/2$  on  $\mathbb{Z}/2 \times \mathbb{Z}/2$  by flipping the components, that is  $(-1)(x, y) := (y, x)$  for all  $x, y \in \mathbb{Z}/2 = \{0, 1\}$ , where again the acting copy of  $\mathbb{Z}/2$  is considered to be the multiplicative group  $\{1, -1\}$ . The Hecke algebra associated to the Hecke pair  $((\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2, \{(0, 0)\} \rtimes \mathbb{Z}/2)$  is commutative, see for instance Proposition 3.5 of [28]. However one can easily check that Condition (3.1) does not hold for this Hecke pair.

The following definition introduces an algebraic notion which implies the relative unimodularity of discrete Hecke pairs:

**Definition 3.3.** Let  $(G, H)$  be a discrete Hecke pair. For every  $g \in G$ , let  $\chi_g$  and  $\chi_{g^{-1}}$  be the characteristic functions of double cosets  $HgH$  and  $Hg^{-1}H$ , respectively. The Hecke pair  $(G, H)$  as well as its Hecke algebra,  $\mathcal{H}(G, H)$ , and its reduced Hecke  $C^*$ -algebra,  $C_r^*(G, H)$ , are called *locally commutative* if  $\chi_g * \chi_{g^{-1}} = \chi_{g^{-1}} * \chi_g$  for all  $g \in G$ .

**Proposition 3.4.** *If  $(G, H)$  is a locally commutative discrete Hecke pair, then it is relatively unimodular.*

*Proof.* For given  $g \in G$ , let  $\chi_g$  and  $\chi_{g^{-1}}$  be as in Definition 3.3. Assume that the double coset  $HgH$  is the disjoint union of right cosets  $Hx_i$  for  $i = 1, \dots, R(g)$ . We compute

$$\begin{aligned} \chi_{g^{-1}} * \chi_g(H) &= \sum_{Hy \in H \backslash G} \chi_{g^{-1}}(Hy^{-1}H) \chi_g(Hy) \\ &= \sum_{i=1}^{R(g)} \chi_{g^{-1}}(Hx_i^{-1}H) = R(g). \end{aligned}$$

By replacing  $g$  with  $g^{-1}$ , we get  $L(g) = \chi_g * \chi_{g^{-1}}(H)$ . The desired statement follows from these equalities.  $\square$

An immediate question is that whether is the converse of the above proposition also true? In other words, if  $\chi_{g^{-1}} * \chi_g(H) = \chi_g * \chi_{g^{-1}}(H)$  for all  $g \in G$ , is it true that  $\chi_{g^{-1}} * \chi_g = \chi_g * \chi_{g^{-1}}$  for all  $g \in G$ ? Clearly every commutative Hecke algebra is locally commutative. Therefore the above proposition applies to Examples 3.1, 3.2 and the following example:

**Example 3.5.** Let  $G = GL_2(\mathbb{Q})^+ := \{g \in GL_2(\mathbb{Q}); \det(g) > 0\}$  and  $H = SL_2(\mathbb{Z})$ . Then  $(G, H)$  is a Hecke pair and its Hecke algebra  $\mathcal{H}(G, H)$  is commutative, see Proposition 1.4.1 and Theorem 1.4.2 of [7]. In fact, the terminology ‘‘Hecke subgroup’’, ‘‘Hecke pair’’ and ‘‘Hecke algebra’’ has been originated from this Hecke pair which was introduced for the first time in the realm of modular forms by E. Hecke, for more historical notes see [17].

Besides above examples, commutative Hecke algebras appear in numerous situations, for instance see Proposition 3.5 of [28].

**Definition 3.6.** A (discrete or non-discrete) Hecke pair  $(G, H)$  is called a *Gelfand pair*, if the Hecke algebra  $\mathcal{H}(G, H)$  is commutative.

An immediate corollary of Equality (2.8) and Proposition 3.4 is:

**Corollary 3.7.** *Let  $G$  be a locally compact group which possesses a compact open subgroup  $H$ . If the discrete Hecke pair  $(G, H)$  is locally commutative, then  $G$  is unimodular.*

**Remark 3.8.** When  $H$  is a normal compact open subgroup of a locally compact group  $G$ , the discrete Hecke pair  $(G, H)$  is always locally commutative, and therefore  $G$  must be unimodular. On the other hand, in this case  $\mathcal{H}(G, H)$  is isomorphic to the complex group algebra of the quotient group  $G/H$ , so it can be a noncommutative algebra. This phenomenon explains why the notion of locally commutative Hecke pairs is more general than the notion of Gelfand pairs.

One notes that the above corollary mostly applies to totally disconnected locally compact groups, because they have plenty of compact open subgroups. In the following we describe two more algebraic conditions implying the relative unimodularity. They come from the work of G.M. Bergman and H.W. Lenstra in [5]. First we need some definitions:

**Definition 3.9.** (i) Two subgroups  $H$  and  $K$  of a group  $G$  are called *commensurable* if  $H \cap K$  is a finite index subgroup of both  $H$  and  $K$ .  
(ii) A subgroup  $H$  of a group  $G$  is called *nearly normal* if it is commensurable with a normal subgroup  $N$  of  $G$ .  
(iii) A subgroup  $H$  of a group is called *almost normal* if it has only finitely many conjugates.

Condition (iii) in the above definition should not be confused with the definition of almost normal subgroups according to [6], see Definition 2.1(ii) and its footnote. One also observes that  $H$  is an almost normal subgroup of  $G$  if and only if the normalizer of  $H$  in  $G$ , is a finite index subgroup of  $G$ .

**Proposition 3.10.** *A discrete Hecke pair  $(G, H)$  is relatively unimodular if one of the following conditions holds:*

- (i) *The subgroup  $H$  is nearly normal in  $G$ .*
- (ii) *The subgroup  $H$  is almost normal in  $G$ .*

*In particular, if a locally compact group  $G$  possesses a compact open subgroup  $H$  which is also almost normal in  $G$ , then  $G$  is unimodular.*

*Proof.* Since the notions defined in Definition 3.9 are algebraic, we do not have to worry about the topology in the following arguments:

- (i) According to Theorem 3 of [5],  $H$  is nearly normal if and only if there exists a natural number  $n$  such that  $1 \leq L(g) \leq n$  for all  $g \in G$ . This statement also follows from the work of G. Schlichting in [22], see Proposition 1(i) of [3]. Thus  $\Delta_{(G, H)}(g) \leq n$  for all  $g \in G$ . On the other hand, since  $\Delta_{(G, H)}$  is a group homomorphism from  $G$  into the multiplicative group  $\mathbb{Q}^+$ , the boundedness of its image amounts to  $\Delta_{(G, H)} = 1$ .
- (ii) It follows from (i) and the fact that if  $H$  is an almost normal and a Hecke subgroup of  $G$ , then it is nearly normal.

□

In [19], B. H. Neumann determined the class of all finitely generated groups all whose subgroups are nearly normal, see also Theorem 2.8 of [25] for a summary. Also, in Example 2.10 of [25], we explained a method to construct nearly normal subgroups of free product of two discrete groups. In its simplest form, it is based on the fact that every finite subgroup of a finitely generated group gives rise to a nearly normal subgroup of a finitely generated free group.

The above discussion suggests a similar study of the unimodularity of locally compact groups. Regarding Assumption 2.7, for a given non-discrete Hecke pair  $(G, H)$ , we are interested in cases where both  $G$  and  $H$  are unimodular.

**Remark 3.11.** Besides Corollary 3.7 and elementary cases where  $H$  is either a discrete, normal, or compact subgroup of a unimodular group  $G$ , see Corollary 1.5.4(a) of [9] and Corollary 2.8 of [11], we have the simultaneous unimodularity of  $G$  and  $H$  in the following cases:

- (i) When  $H$  is a lattice in  $G$ . A discrete subgroup  $H$  of a locally compact group  $G$  is called a *lattice in  $G$*  if the homogeneous space  $H \backslash G$  has a  $G$ -invariant Radon measure  $\nu$  such that  $\nu(H \backslash G) < \infty$ . In this case, by Theorem 9.1.6 of [9],  $G$  is unimodular too.
- (ii) When  $H$  is a unimodular and cocompact subgroup of a locally compact group  $G$ . In this case, by Proposition 9.1.2 of [9],  $G$  is unimodular too. In fact the same conclusion is valid even when  $H \backslash G$  has a finite relatively invariant measure. A measure  $\nu$  on  $H \backslash G$  is called *relatively invariant* if there exists a function  $\chi : G \rightarrow ]0, \infty[$  such that  $\nu(Eg) = \chi(g)\nu(E)$  for all  $g \in G$  and for all measurable subsets  $E \subseteq H \backslash G$ . In this case the function  $\chi$  is a continuous homomorphism from  $G$  into  $\mathbb{R}^+$  and is called the *character of  $\nu$* . When  $H$  is unimodular, there is a relatively invariant measure on  $H \backslash G$ , which is unique up to a positive multiple, and moreover, if  $\nu(H \backslash G) < \infty$ , then  $G$  is unimodular too, see Corollary B.1.8 of [4].
- (iii) When  $G$  is unimodular and  $H$  is a nearly normal subgroup of  $G$ . Let  $N$  be a normal subgroup of  $G$  commensurable with  $H$ . If  $N$  is not closed replace it with its closure, which is still commensurable with  $H$ . Then the statement follows from Lemma 3.12 below.

**Lemma 3.12.** *Let  $H$  and  $K$  be two commensurable closed subgroups of a locally compact group  $G$ . Then  $H$  is unimodular if and only if  $K$  is unimodular.*

*Proof.* Without loss of generality, we can assume that  $K$  is a finite index subgroup of  $H$ . By replacing  $K$  with  $N := \bigcap_{h \in H} hKh^{-1}$ , we can even assume  $K$  is normal. Then the statement follows from Remark 3.11(ii). □

#### 4. WHEN $H$ IS A COMPACT SUBGROUP OF $G$

In this section we show that when  $H$  is a compact subgroup of a locally compact group  $G$ , the pair  $(G, H)$  is a Hecke pair. We also discuss briefly some classes of examples of such pairs. The special case where  $H$  is a compact and open subgroup of  $G$  will be treated with more details in the next subsection, because it is better studied within the class of Hecke pairs with open Hecke subgroups.

Define a map

$$(4.1) \quad \iota : C_c(H \backslash G) \rightarrow C_c(G),$$

$$\iota(f)(x) = \tilde{f}(x) := f(Hx), \quad \forall f \in C_c(H \backslash G), x \in G.$$

Since  $H$  is compact, this map is well defined. We also consider its extensions from  $L^2(H \backslash G)$  into  $L^2(G)$  and from  $L^1(H \backslash G)$  into  $L^1(G)$ , and denote them with the same notation. Since  $\tilde{f}$  is left  $H$ -invariant, one checks that

$$\int_H |\tilde{f}(hx)|^2 d\eta(h) = |\tilde{f}(x)|^2 \int_H d\eta(h) = \eta(H) |f(Hx)|^2, \quad \forall x \in G.$$

Thus it follows from Weil's formula, (2.5), that  $\|\tilde{f}\|_2^2 = \eta(H) \|f\|_2^2$  for all  $f \in L^2(H \backslash G)$  where the  $L^2$ -norms of  $\tilde{f}$  and  $f$  are computed in  $L^2(G)$  and  $L^2(H \backslash G)$ , respectively. Similarly, we have  $\|\tilde{f}\|_1 = \eta(H) \|f\|_1$ . One also notes that for every  $f \in \mathcal{H}(G, H)$ ,  $\tilde{f}$  is a bi- $H$ -invariant function on  $G$ , and so  $f(Hx) = \tilde{f}(xh)$  for all  $x \in G$  and  $h \in H$ . Therefore for every  $f \in \mathcal{H}(G, H)$ ,  $g \in L^2(H \backslash G)$  and  $x \in G$ , we can compute

$$\begin{aligned} \widetilde{f * g}(x) &= f * g(Hx) = \int_{H \backslash G} f(Hxy^{-1}) g(Hy) d\nu(y) \\ &= \frac{1}{\eta(H)} \int_{H \backslash G} \tilde{f}(xy^{-1}) \tilde{g}(y) \left( \int_H d\eta(h) \right) d\nu(y) \\ &= \frac{1}{\eta(H)} \int_{H \backslash G} \left( \int_H \tilde{f}(xy^{-1}h^{-1}) \tilde{g}(hy) d\eta(h) \right) d\nu(y) \\ &= \frac{1}{\eta(H)} \int_G \tilde{f}(xy^{-1}) \tilde{g}(y) d\mu(y) \\ &= \frac{1}{\eta(H)} \tilde{f} * \tilde{g}(x). \end{aligned}$$

Hence

$$\begin{aligned} \|f * g\|_2^2 &= \frac{1}{\eta(H)} \|\widetilde{f * g}\|_2^2 = \frac{1}{\eta(H)^3} \|\tilde{f} * \tilde{g}\|_2^2 \\ &\leq \frac{1}{\eta(H)^3} \|\tilde{f}\|_1^2 \|\tilde{g}\|_2^2 = \|f\|_1^2 \|g\|_2^2. \end{aligned}$$

We summarize the above computations in the following theorem:

**Theorem 4.1.** *Let  $H$  be a compact subgroup of a locally compact group  $G$ . Then the pair  $(G, H)$  is a Hecke pair. Furthermore, the left regular representation  $\lambda : \mathcal{H}(G, H) \rightarrow B(L^2(H \backslash G))$  is bounded and we have  $\|\lambda(f)\| \leq \|f\|_1$ .*

One notes that the above theorem is valid particularly when  $H$  is a compact open subgroup of  $G$ , whether the Hecke pair  $(G, H)$  is considered as a discrete or non-discrete Hecke pair. In fact, this is where two settings of locally compact Hecke pairs and discrete Hecke pairs coincide with each other, and it allows us to transfer many results and ideas from one setting to another.

**Remark 4.2.** When  $(G, H)$  is as the above theorem, we define  $L^1(G//H)$  as the involutive Banach algebra consisting of those functions in  $L^1(H \backslash G)$  that are almost everywhere right

$H$ -invariant. Then by the continuity of  $\lambda$ , we can extend the left regular representation to whole  $L^1(G//H)$ .

Now we discuss several examples of Hecke pairs  $(G, H)$  satisfying the assumption of Theorem 4.1. These examples have already studied in other branches of mathematics such as noncommutative harmonic analysis, representations theory, Lie theory, geometric group theory and random walks. Therefore not only we have no shortage of examples for the above theorem, but we also have plenty of opportunities to find new notable applications for our extended  $C^*$ -algebraic formulation of Hecke algebras.

When  $H$  is a finite subgroup of  $G$ , no matter  $G$  is discrete or not, one can apply Theorem 4.1 to show that the left regular representation is a bounded operator, see Examples 3.1(ii) and 3.2.

Compact Lie subgroups of Lie groups give rise to another class of examples. We are particularly interested in an important subclass of these Hecke pairs: The class consisting of pairs  $(G, H)$ , where  $H$  is a maximal compact subgroup of a Lie group  $G$ , see [14]. This subclass is big enough to contain Gelfand pairs  $(G, H)$  which appear in Lie theory and representation theory.

**Example 4.3.** In the following we give some examples of the Hecke pairs  $(G, H)$  described in the above.

- (i) For every  $n \geq 2$ ,  $SO_n(\mathbb{R})$  is a maximal compact subgroup of  $SL_n(\mathbb{R})$  and the Hecke pair  $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$  is a Gelfand pair. For a quick reference of these facts for the case  $n = 2$ , and without using Lie theory, see Theorem 11.1.1 and Theorem 11.2.3 of [9].
- (ii) Every closed subgroup of a compact Lie group is an example too. For instance, consider a maximal connected abelian subgroup  $H$  of a compact Lie group  $G$ . In this case,  $H$  is called a *maximal torus* of  $G$  and is isomorphic to  $\mathbb{T}^k$  for some natural number  $1 \leq k \leq \dim(G)$ , see [14, 23] for their existence, various properties and concrete examples. One notes that these Hecke pairs are also examples for Hecke pairs that will be discussed in Section 6, and particularly in Remark 6.3(ii).

We can extend the above classes of Hecke pairs further by considering pro-Lie groups. A locally compact group  $G$  is called *pro-Lie group* if there exists a small compact normal subgroup  $K$  of  $G$  such that  $G/K$  is a Lie group, see [1] for equivalent definitions. Then every (maximal) compact subgroup of  $G/K$  gives rise to a (maximal) compact subgroup  $H$  of  $G$  and it follows from Proposition 2.13 that the Hecke algebra of  $\mathcal{H}(G, H)$  is isomorphic to the Hecke algebra  $(G/K, H/K)$ . One notes that  $K$  does not need to be compact here, as long as it is normal in  $G$  and  $G/K$  is a Lie group. One can find more similar situations in the literature, for instance, see Theorem 2 of [18] and Theorem 5 of [20].

Another class of examples of Hecke pairs satisfying the assumptions of Theorem 4.1 consists of Hecke pairs  $(G, H)$ , where  $G$  is a totally disconnected locally compact group and  $H$  is a compact (and in most interesting cases, open) subgroup of  $G$ . Many concrete examples of this type arise naturally as the automorphism groups of graphs (especially trees) and compact subgroups of them. Here we content ourself with examples of compact subgroups of the automorphism groups of homogeneous trees, see [10] for details.

**Example 4.4.** Given a homogeneous tree  $\mathfrak{X}$ , let the same notation  $\mathfrak{X}$  denote the set of its vertices and let  $\mathfrak{E}$  denote the set its edges. The tree  $\mathfrak{X}$  has a natural metric structure and the group  $\text{Aut}(\mathfrak{X})$  of all isometries of  $\mathfrak{X}$  with respect to this metric equipped with the

permutation topology is a totally disconnected locally compact group. It is well known that  $Aut(\mathfrak{X})$  possesses plenty of lattices, see for instance [2]. Thus, by Remark 3.11(i),  $Aut(\mathfrak{X})$  is unimodular, and consequently, for every compact open subgroup  $H$  of  $Aut(\mathfrak{X})$ , the discrete Hecke pair  $(Aut(\mathfrak{X}), H)$  is relatively unimodular. For every  $x \in \mathfrak{X}$  (resp.  $\{x, y\} \in \mathfrak{E}$ ), let  $K_x$  (resp.  $K_{\{x, y\}}$ ) denote the subgroup of  $Aut(\mathfrak{X})$  which stabilizes the vertex  $x$  (resp. the edge  $\{x, y\}$ ). There are certain characterizations and examples of compact subgroups of  $Aut(\mathfrak{X})$  as follows.

- (i) For given  $x \in \mathfrak{X}$  and  $\{y, z\} \in \mathfrak{E}$ , the subgroups  $K_x$  and  $K_{\{y, z\}}$  are compact open subgroups of  $Aut(\mathfrak{X})$ . In fact, by Theorem 5.2 of [10], subgroups of these forms are maximal compact subgroups of  $\mathfrak{X}$ .
- (ii) Given a closed subgroup  $K$  of  $Aut(\mathfrak{X})$ ,  $K$  is compact if and only if  $K(x)$ , the orbit of  $x$  under the action of  $K$ , is finite for all  $x \in \mathfrak{X}$ , see Proposition 5.1 of [10].
- (iii) Let  $\gamma = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$  be a geodesic in  $\mathfrak{X}$ . Define  $M_\gamma := \bigcap_{i=-\infty}^{\infty} K_{x_i}$ . Then  $M_\gamma$  is a compact subgroup of  $Aut(\mathfrak{X})$  and it is not open in  $Aut(\mathfrak{X})$ .

The above examples of Hecke pairs are particularly important to find examples of Hecke pairs with property (RD), see Example 3.17 of [27]. Yet there is another important class of Hecke pairs of the type described in Theorem 4.1. This class consists of the Schlichting completions of reduced discrete Hecke pairs  $(G, H)$ . Since the subgroup in a Schlichting completion is open, we study these examples in the next section.

## 5. WHEN $H$ IS AN OPEN SUBGROUP OF $G$

In this section  $(G, H)$  is always a discrete Hecke pair. Using the Schlichting completion of  $(G, H)$ , we reduce the study of this type of Hecke pairs to the case that the subgroup in the Hecke pair is compact and open.

**Definition 5.1.** Given a discrete Hecke pair  $(G, H)$ , the normal subgroup of  $G$  defined by  $K_{(G, H)} := \bigcap_{x \in G} xHx^{-1}$  is called the *core of the Hecke pair*  $(G, H)$ . The Hecke pair  $(G, H)$  is called *reduced* if its core is the trivial subgroup. The pair  $(\frac{G}{K_{(G, H)}}, \frac{H}{K_{(G, H)}})$  is a reduced discrete Hecke pair which is called the *reduction of*  $(G, H)$  or the *reduced Hecke pair associated with*  $(G, H)$  and is denoted by  $(G_r, H_r)$ .

The reduced Hecke pair  $(G_r, H_r)$  associated with  $(G, H)$  has a similar properties as  $(G, H)$  in a number of situations, for instance see Lemmas 3.11 and 4.2 of [27]. In fact, it follows from Proposition 2.13 that the Hecke algebras of Hecke pairs  $(G, H)$  and  $(G_r, H_r)$  are naturally isomorphic. Therefore we often assume that discrete Hecke pairs are reduced.

The Schlichting completion is a process to associate a totally disconnected locally compact group  $\overline{G}$  with a given reduced discrete Hecke pair  $(G, H)$  such that  $G$  is dense in  $\overline{G}$ , and more importantly, the closure of  $H$  in  $\overline{G}$ , denoted by  $\overline{H}$ , is a compact open subgroup of  $\overline{G}$ . Then the discrete Hecke pair  $(\overline{G}, \overline{H})$  is called the *Schlichting completion of*  $(G, H)$ . We refer the reader to [16] for a detailed account of the Schlichting completion. The following examples explains some aspects of the subject.

**Example 5.2.** Let  $(G, H)$  be a discrete Hecke pair.

- (i) Assume that  $(G, H)$  is reduced. Then it follows that the Schlichting completion of  $(G, H)$  is the same as  $(G, H)$  if and only if  $H$  is a finite group.
- (ii) Let  $H$  be a nearly normal subgroup of  $G$ . If  $H$  is finitely generated, then the subgroup  $H_r$  in the reduced Hecke pair  $(G_r, H_r)$  associated to  $(G, H)$  is finite, and



consequently we have  $(\overline{G}_r, \overline{H}_r) = (G_r, H_r)$ . To prove this, it is enough to show that  $H$  has a finite index subgroup  $L$  which is normal in  $G$ . Let  $N$  be a normal subgroup of  $G$  which is commensurable with  $H$ . Since  $H$  is finitely generated,  $N$  is finitely generated too, and therefore there are only finitely many subgroups of index equal to  $[N : H \cap N]$  in  $N$ . Set

$$L := \cap_{x \in G} x(H \cap N)x^{-1}.$$

Then  $L$  is finite index in  $H$  and normal in  $G$ . This proves the above claim. I have learnt this argument from Henry Wilton in mathoverflow.net.

When  $H$  is not finitely generated, the Hecke pair  $(G, H)$  can be reduced, and consequently its Schlichting completion is different than itself. For example, let  $N$  be an infinite product of  $\mathbb{Z}/2\mathbb{Z}$  and let  $A$  be the full automorphism group of  $N$ . Set  $G := N \rtimes A$  and let  $H$  be a subgroup of  $N$  of index 2. Then since  $A$  acts transitively on nontrivial elements of  $N$ ,  $H$  does not contain any non-trivial normal subgroup of  $G$ . I have learnt this example from Jeremy Rickard in mathoverflow.net.

One notes that when  $H$  is a finite index subgroup of  $G$ , the condition of  $H$  being finitely generated is unnecessary. Because, in any case, the subgroup  $K_{(G,H)}$  is a finite index subgroup of  $H$ . Therefore we have  $(\overline{G}_r, \overline{H}_r) = (G_r, H_r)$ .

- (iii) Let  $(G, H)$  be a the Bost-Connes Hecke pair discussed in Example 2.4. One checks that it is a reduced Hecke pair and it is shown in Example 11.4 of [16] that its Schlichting completion  $(\overline{G}, \overline{H})$  is as follows:

$$\overline{G} = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} ; a \in \mathbb{Q}^+, b \in \mathcal{A} \right\}, \quad \text{and} \quad \overline{H} = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} ; r \in \mathcal{Z} \right\},$$

where  $\mathcal{A}$  and  $\mathcal{Z}$  are the ring of finite adeles on  $\mathbb{Q}$  and its the maximal compact subring, respectively. Moreover, the topology of the above Schlichting completion coincides with the topology coming from the topology of  $\mathcal{A}$ , the restricted product topology of  $p$ -adic fields.

- (iv) Given a prime number  $p$ , consider the Hecke pair  $(SL_2(\mathbb{Z}[1/p]), SL_2(\mathbb{Z}))$ . By looking at intersections of the form  $SL_2(\mathbb{Z}) \cap x_n SL_2(\mathbb{Z}) x_n^{-1}$ , where  $x_n = \begin{pmatrix} 0 & p^{-n} \\ -p^n & 0 \end{pmatrix}$  for  $n \in \mathbb{N}$ , one easily observes that  $K_{(SL_2(\mathbb{Z}[1/p]), SL_2(\mathbb{Z}))} = \{I, -I\}$ , where  $I$  is the  $2 \times 2$  identity matrix. Therefore the pair  $(PSL_2(\mathbb{Z}[1/p]), PSL_2(\mathbb{Z}))$  is the reduction of the Hecke pair  $(SL_2(\mathbb{Z}[1/p]), SL_2(\mathbb{Z}))$ . Then it is shown in Example 11.8 of [16] that the Schlichting completion of  $(PSL_2(\mathbb{Z}[1/p]), PSL_2(\mathbb{Z}))$  is  $(PSL_2(\mathbb{Q}_p), PSL_2(\mathbb{Z}_p))$ , which is the reduction of the Hecke pair appearing in Example 3.1(i). Similar discussion shows that  $(PSL_2(\mathcal{A}), PSL_2(\mathcal{Z}))$  is the Schlichting completion of  $(PSL_2(\mathbb{Q}), PSL_2(\mathbb{Z}))$ . One notes that the Hecke topology of these Schlichting completions again coincide with the conventional totally disconnected locally compact topologies of the above groups coming from  $p$ -adic valuations and the restricted product of  $p$ -adic fields.

The following lemma simply states that passing to the Schlichting completion does not change the algebraic and analytic aspects of reduced discrete Hecke pairs and their associated Hecke algebras.

**Lemma 5.3.** ([16], Proposition 4.9) *Let  $(G, H)$  be a reduced discrete Hecke pair and let  $(\overline{G}, \overline{H})$  be its Schlichting completion. Then the following statements hold:*

- (i) The mapping  $\alpha : H \backslash G \rightarrow \overline{H} \backslash \overline{G}$  (resp.  $\alpha' : G/H \rightarrow \overline{G}/\overline{H}$ ), defined by  $Hg \mapsto \overline{H}g$  (resp.  $gH \mapsto g\overline{H}$ ) for all  $g \in G$  is a  $G$ -equivariant bijection. In particular,  $\alpha$  induces an isometric isomorphism between Hilbert spaces  $\ell^2(H \backslash G)$  and  $\ell^2(\overline{H} \backslash \overline{G})$ .
- (ii) The mapping  $\beta : G//H \rightarrow \overline{G}//\overline{H}$ , defined by  $HgH \mapsto \overline{H}g\overline{H}$  for all  $g \in G$  is a bijection.
- (iii) The mapping  $\beta$  commutes with the convolution product, and therefore it induces an isometric isomorphism between the Hecke algebras  $\mathcal{H}(G, H)$  and  $\mathcal{H}(\overline{G}, \overline{H})$ , with respect to the corresponding  $\ell^1$ -norms.

The next theorem is an important application of the Schlichting completion. For other applications see [28, 16, 26, 27].

**Theorem 5.4.** *Let  $(G, H)$  be a discrete Hecke pair. Then the left regular representation  $\lambda : \mathcal{H}(G, H) \rightarrow B(\ell^2(H \backslash G))$  is bounded. In fact, we have  $\|\lambda(f)\| \leq \|f\|_1$  for all  $f \in \mathcal{H}(G, H)$ .*

*Proof.* By Proposition 2.13, without loss of generality, we can assume  $(G, H)$  is a reduced discrete Hecke pair. Let  $\alpha$  and  $\beta$  be as Lemma 5.3. Since both Hecke pairs  $(G, H)$  and  $(\overline{G}, \overline{H})$  are discrete, the above mappings induce the following isometric isomorphisms:

$$\begin{aligned} \alpha^* : \ell^2(\overline{H} \backslash \overline{G}) &\rightarrow \ell^2(H \backslash G), & \alpha^*(\xi) &:= \xi \circ \alpha, & \forall \xi \in \ell^2(\overline{H} \backslash \overline{G}), \\ \beta^* : \mathcal{H}(\overline{G}, \overline{H}) &\rightarrow \mathcal{H}(G, H), & \beta^*(f) &:= f \circ \beta, & \forall f \in \mathcal{H}(\overline{G}, \overline{H}), \end{aligned}$$

where the norm on Hecke algebras  $\mathcal{H}(G, H)$  and  $\mathcal{H}(\overline{G}, \overline{H})$  are the corresponding  $\ell^1$ -norms. One notes that a similar isometry as  $\alpha^*$  exists from  $\ell^1(\overline{H} \backslash \overline{G})$  onto  $\ell^1(H \backslash G)$ . It is straightforward to check that this isometries commute with the convolution products defining the left regular representations of  $\mathcal{H}(G, H)$  and  $\mathcal{H}(\overline{G}, \overline{H})$ . Since  $\overline{H}$  is compact, by applying Theorem 4.1 and the above discussion for every  $f \in \mathcal{H}(G, H)$  and  $\xi \in \ell^2(H \backslash G)$ , we have

$$\begin{aligned} \|f * \xi\|_2 &= \|\alpha^*((\beta^{*-1}f) * (\alpha^{*-1}\xi))\|_2 \\ &= \|(\beta^{*-1}f) * (\alpha^{*-1}\xi)\|_2 \\ &\leq \|\beta^{*-1}f\|_1 \|\alpha^{*-1}\xi\|_2 \\ &= \|f\|_1 \|\xi\|_2. \end{aligned}$$

□

It worths mentioning that the above theorem is stronger than Proposition 1.3.3 of [8], and more importantly, it is a necessary step to show that Hecke pairs of polynomial growth possess property (RD), see Proposition 4.8(i) of [27].

## 6. WHEN $H$ IS A COCOMPACT SUBGROUP OF $G$

In this section, we assume that  $H$  is a unimodular closed subgroup of  $G$  such that the homogeneous space  $H \backslash G$  has a finite relatively invariant measure  $\nu$ . Therefore by Remark 3.11(ii),  $G$  is unimodular too, and so both Assumptions 2.2 and 2.7 hold for the pair  $(G, H)$ .

**Theorem 6.1.** *Given a pair  $(G, H)$  as above, the left regular representation  $\lambda : \mathcal{H}(G, H) \rightarrow B(L^2(H \backslash G))$  is a well defined  $*$ -homomorphism.*

*Proof.* For a given  $f \in \mathcal{H}(G, H)$ , we set  $M_f := \max\{|f(Hx)|; Hx \in H \backslash G\}$ . Then for every  $\xi \in L^2(H \backslash G)$ , by applying Minkowski's inequality for integrals (Theorem 6.19 of [12]), we

obtain

$$\begin{aligned}
 \|f * \xi\|_2 &= \left( \int_{H \backslash G} \left| \int_{H \backslash G} f(Hxy^{-1}) \xi(Hy) d\nu(Hy) \right|^2 d\nu(Hx) \right)^{1/2} \\
 &\leq \int_{H \backslash G} \left( \int_{H \backslash G} |f(Hxy^{-1})|^2 |\xi(Hy)|^2 d\nu(Hy) \right)^{1/2} d\nu(Hx) \\
 &\leq M_f \int_{H \backslash G} \|\xi\|_2 d\nu(Hx) \\
 &= M_f \nu(H \backslash G) \|\xi\|_2.
 \end{aligned}$$

This shows that  $f * \xi \in L^2(H \backslash G)$ . □

The following corollary follows from the above theorem and Remark 2.11(iii):

**Corollary 6.2.** *If  $H$  is a unimodular cocompact subgroup of a locally compact group  $G$ , then the pair  $(G, H)$  is a Hecke pair.*

In the following we describe two cases of Hecke pairs  $(G, H)$  as described in the above corollary and for which the left regular representation  $\lambda$  is bounded.

- Remark 6.3.** (i) If  $H$  is a finite index closed subgroup of  $G$ , then  $H$  is also open, and so the Hecke pair  $(G, H)$  is a discrete Hecke pair. Thus by Theorem 5.4,  $\lambda$  is bounded.
- (ii) When  $H$  contains a normal cocompact subgroup  $K$  of  $G$ . In this case the quotient groups  $G/K$  and  $H/K$  are both compact. Therefore by Proposition 2.13 and Theorem 4.1,  $\lambda$  is bounded.

One notes that not every finite index subgroup of a locally compact group is closed, so this condition is necessary in Remark 6.3(i). In Proposition 3.3 of [27], it is shown how the boundedness of  $\lambda$  in the above cases implies property (RD).

We do not know whether pairs  $(G, H)$  for which  $H \backslash G$  has a finite relatively invariant measure satisfy Condition (b) of Definition 2.10(i). Regarding the above discussion, one notes that this condition is the only thing potentially prevents these pairs to be Hecke pairs.

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*E-mail address:* shirbisheh@gmail.com